An Extension of the Mode Theory to Periodically Distributed Parametric Amplifiers with Losses*

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Summary—For the extension of the mode theory of the lossless periodically distributed parametric amplifier to the lossy case, a "conjugate circuit" is introduced in this paper. The conjugate circuit is an imaginary circuit which is obtained in the pass band by replacing each resistance in the original circuit with the negative resistance of the same magnitude. The orthogonality properties between the modes of the original circuit and those of the conjugate circuit are derived. The power gain and the noise figure of the amplifier are calculated, showing the usefulness of this mode theory in accounting for the spreading resistance of the semiconductor diode.

Introduction

In a previous paper,[1] an operator T_θ was introduced for the analysis of the lossless periodically distributed parametric amplifiers. The operator T_θ is the product of a diagonal matrix expressing the pumping phase relation and the T matrix of the basic section of the amplifier. The eigenvectors of T_θ are called the modes of the amplifier. The orthogonality properties among the modes were proved, using one of the Manley-Rowe relations. However, the proof required that the circuit be lossless. In practical amplifiers the spreading resistance of the semiconductor diode cannot be neglected. Therefore, it is desired to obtain some substitutes for the orthogonal properties among the modes. For this purpose, a "conjugate circuit" is introduced in this paper. The orthogonality properties which exist between the modes of the original circuit and those of the conjugate circuit are proved, thereby showing that the mode theory remains useful even when the spreading resistance and the other losses of the circuit are taken into account.

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[1] K. Kurokawa and J. Hamasaki, "Mode theory of lossless periodically distributed parametric amplifiers," IRE Trans. on Microwave Theory and Techniques, vol. MTT-7, pp. 360–365; July, 1959.

The Operator T_θ

The equivalent circuit of the semiconductor diode is shown in Fig. 1. In this figure, r represents the spreading resistance, which is assumed to be independent of the frequency.

The relation between the current i flowing into the diode and the applied voltage v is

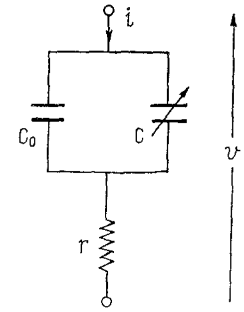

Fig. 1—Equivalent circuit of semiconductor diode.

$$\begin{bmatrix} i_1 \\ i_2^* \end{bmatrix} = \begin{bmatrix} j\omega_1 c_0, & j\omega_1 \dfrac{c}{2} \\ -j\omega_2 \dfrac{c^*}{2}, & -j\omega_2 c_0 \end{bmatrix} \begin{bmatrix} v_1 - ri_1 \\ v_2^* - ri_2^* \end{bmatrix} \tag{1}$$

where the subscripts 1 and 2 refer to the angular frequencies ω_1 and ω_2, respectively, and the pumping angular frequency ω_p is assumed to be $\omega_1+\omega_2$. Rewriting (1) in the form

$$\left\{ I + r \begin{bmatrix} j\omega_1 c_0, & j\omega_1 \dfrac{c}{2} \\ -j\omega_2 \dfrac{c^*}{2}, & j\omega_2 c_0 \end{bmatrix} \right\} \begin{bmatrix} i_1 \\ i_1^* \end{bmatrix} = \begin{bmatrix} j\omega_1 c_0, & j\omega_1 \dfrac{c}{2} \\ -j\omega_2 \dfrac{c^*}{2}, & -j\omega_2 c_0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2^* \end{bmatrix} \tag{2}$$

where I is a unit matrix, and then multiplying by

$$\left\{ I + r \begin{bmatrix} j\omega_1 c_0, & j\omega_1 \dfrac{c}{2} \\ -j\omega_2 \dfrac{c^*}{2}, & -j\omega_2 c_0 \end{bmatrix} \right\}^{-1}$$

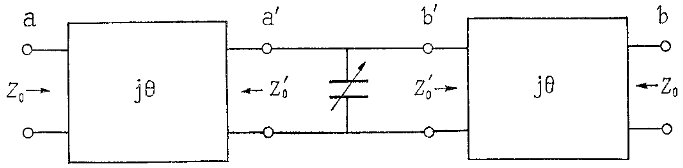

Fig. 2—Basic section of the amplifier.

from the left, we have

$$\begin{bmatrix} i_1 \\ i_2^* \end{bmatrix} \doteqdot \left\{ I - r \begin{bmatrix} j\omega_1 c_0, & j\omega_1 \dfrac{c}{2} \\ -j\omega_2 \dfrac{c^*}{2}, & -j\omega_2 c_0 \end{bmatrix} \right\} \begin{bmatrix} j\omega_1 c_0, & j\omega_1 \dfrac{c}{2} \\ -j\omega_2 \dfrac{c^*}{2}, & -j\omega_2 c_0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2^* \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} j\omega_1 c_0, & j\omega_1 \dfrac{c}{2} \\ -j\omega_2 \dfrac{c^*}{2}, & -j\omega_2 c_0 \end{bmatrix} - r \begin{bmatrix} -\omega_1^2 c_0^2 + \omega_1\omega_2 \dfrac{|c|^2}{4}, & -\omega_1^2 \dfrac{c_0 c}{2} + \omega_1\omega_2 \dfrac{c c_0}{2} \\ \omega_1\omega_2 \dfrac{c^* c_0}{2} - \omega_2^2 \dfrac{c_0 c^*}{4}, & \omega_1\omega_2 \dfrac{|c|^2}{4} - \omega_2^2 c_0^2 \end{bmatrix} \right\} \begin{bmatrix} v_1 \\ v_2^* \end{bmatrix}. \quad (3)$$

In (3) we neglect the higher order terms of r which are assumed to be small. The c_0 in the first matrix on the right hand side of (3) is divided into two parts, each of which is included in the two terminal pair networks (a) and (b) of the basic section of the amplifier, which is shown in Fig. 2. The relation between the current I_c flowing into the remaining circuit (c) and the applied voltage V_c is

$$I_c = \begin{bmatrix} I_1 \\ I_2^* \end{bmatrix} = \begin{bmatrix} r\left(\omega_1^2 c_0^2 - \omega_1\omega_2 \dfrac{|c|^2}{4}\right), & j\omega_1 \dfrac{c}{2}\{1 + j(\omega_2 - \omega_1)c_0 r\} \\ -j\omega_2 \dfrac{c^*}{2}\{1 + j(\omega_2 - \omega_1)c_0 r\}, & r\left(\omega_2^2 c_0^2 - \omega_1\omega_2 \dfrac{|c|^2}{4}\right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2^* \end{bmatrix}. \quad (4)$$

The voltage and current at each terminal in Fig. 2 are, in terms of the incident waves (subscript i) and the reflected waves (subscript r),

$$\begin{aligned} V_a &= \sqrt{Z_0}(a_i + a_r), & V_a' &= \sqrt{Z_0'}(a_i e^{-j\theta} + a_r e^{j\theta}) \\ I_a &= \frac{1}{\sqrt{Z_0}}(a_i - a_r), & I_a' &= \frac{1}{\sqrt{Z_0'}}(a_i e^{-j\theta} - a_r e^{j\theta}) \\ V_b &= \sqrt{Z_0}(b_i + b_r), & V_b' &= \sqrt{Z_0'}(b_i e^{j\theta} + b_r e^{-j\theta}) \\ I_b &= \frac{1}{\sqrt{Z_0}}(b_i - b_r), & I_b' &= \frac{1}{\sqrt{Z_0'}}(b_i e^{j\theta} - b_r e^{-j\theta}) \quad (5) \end{aligned}$$

where the prime notation indicates the values of the (c) side terminals. Since the voltages at the (c) terminals must be equal,

$$V_a' = V_b' = V_c. \quad (6)$$

The equation of continuity is

$$I_a' = I_b' + I_c. \quad (7)$$

Using (4)–(7), we obtain the relation

$$B = TA \quad (8)$$

between the input vector A and the output vector B of the basic section, where

$$A = \begin{bmatrix} a_{i1} \\ a_{r1} \\ a_{i2}^* \\ a_{r2}^* \end{bmatrix}, \qquad B = \begin{bmatrix} b_{i1} \\ b_{r1} \\ b_{i2}^* \\ b_{r2}^* \end{bmatrix}, \quad (9)$$

$$T = \begin{bmatrix} (1 - \sigma_1)e^{-2j\theta_1}, & -\sigma_1, & -j\omega_1 c Z e^{j(\theta_2^* - \theta_1)}, & -j\omega_1 c Z e^{-j(\theta_1 + \theta_2^*)} \\ \sigma_1, & (1 + \sigma_1)e^{2j\theta_1}, & j\omega_1 c Z e^{j(\theta_1 + \theta_2^*)}, & j\omega_1 c Z e^{j(\theta_1 - \theta_2^*)} \\ j\omega_2 c^* Z e^{j(\theta_2^* - \theta_1)}, & j\omega_2 c^* Z e^{j(\theta_1 + \theta_2^*)}, & (1 - \sigma_2)e^{2j\theta_2^*}, & -\sigma_2 \\ -j\omega_2 c^* Z e^{-j(\theta_1 + \theta_2^*)}, & -j\omega_2 c^* Z e^{j(\theta_1 - \theta_2^*)}, & \sigma_2, & (1 + \sigma_2)e^{-2j\theta_2^*} \end{bmatrix}, \quad (10)$$

and

$$\sigma_1 = \frac{Z_{01}'}{2} r \left(\omega_1^2 c_0^2 - \omega_1 \omega_2 \frac{|c|^2}{4} \right)$$

$$\sigma_2 = \frac{Z_{02}'^*}{2} r \left(\omega_2^2 c_0^2 - \omega_1 \omega_2 \frac{|c|^2}{4} \right)$$

$$Z = \frac{\sqrt{Z'_{01} Z_{02}'^*}}{4} \{1 + j(\omega_2 - \omega_1) c_0 r\}. \tag{11}$$

Next, we shall consider the n similar circuits connected in cascade. The variable capacitor of each circuit has the pumping phase lagged by $2\theta_p$ (θ_p: real) from that of the preceding one. These circuits are represented by matrices similar to T except they have $ce^{-2j\theta_p}$, $ce^{-4j\theta_p} \cdots$ in place of c. The pumping phase does not appear in σ_1, σ_2 and Z. Thus we can write the T matrix of the kth section in the form

$$I_\theta^{k-1} T I_\theta^{-(k-1)}$$

where

$$I_\theta = \begin{bmatrix} e^{-j\theta_p} & 0 & 0 & 0 \\ 0 & e^{-j\theta_p} & 0 & 0 \\ 0 & 0 & e^{j\theta_p} & 0 \\ 0 & 0 & 0 & e^{j\theta_p} \end{bmatrix}. \tag{12}$$

The output vector of the amplifier is, therefore,

$$B = (I_\theta^{n-1} T I_\theta^{-n+1})(I_\theta^{n-2} T I_\theta^{-n+2}) \cdots (I_\theta T I_\theta^{-1}) T A$$
$$= I_\theta^n T_\theta^n A \tag{13}$$

where A is the input vector and

$$T_\theta = I_\theta^{-1} T \tag{14}$$

as in the previous paper.

Conjugate Circuit and Orthogonality Properties

From (13), we see that each eigenvector A_k of T_θ is independently transformed by the amplifier into $\lambda_k^n I_\theta^n A_k$, where λ_k is the eigenvalue of A_k. This is the result which we first obtained in the lossless case. In the lossless case, we had another useful result: there are orthogonality properties between the modes. Naturally, we wish to prove them. The proof requires, however, that the circuit be lossless, and this condition is not satisfied in the present case. Without the orthogonality properties, the mode theory loses most of its power. To get the substitutes, we shall introduce a conjugate circuit.[2]

The conjugate circuit is an imaginary circuit with θ_1^*, θ_2^*, $Z_{01}'^*$, $Z_{02}'^*$ and $-r$ in place of θ_1, θ_2, Z_{01}', Z_{02}' and r, respectively, of the original circuit. Assuming that the losses are small, we can obtain the conjugate circuit in the pass band of ω_1 and ω_2 by replacing each resistance in the original circuit by the negative resistance of the same magnitude. We shall use the symbol $\sim$ to indicate the complex conjugate transpose of the corresponding matrix for the conjugate circuit.

The T matrix of the basic section of the conjugate circuit is the matrix with θ_1^*, θ_2, $-\sigma_1^*$, $-\sigma_2^*$ and Z^* in place of θ_1, θ_2^*, σ_1, σ_2 and Z, respectively, of the T matrix given by (10). Its complex conjugate transposed matrix is

$$\tilde{T} = \begin{bmatrix} (1+\sigma_1)e^{2j\theta_1}, & -\sigma_1, & -j\omega_2 c Z e^{j(\theta_1-\theta_2^*)}, & j\omega_2 c Z e^{j(\theta_1+\theta_2^*)} \\ \sigma_1 & (1-\sigma_1)e^{-2j\theta_1}, & -j\omega_2 c Z e^{-j(\theta_1+\theta_2^*)}, & j\omega_2 c Z e^{j(\theta_2^*-\theta_1)} \\ j\omega_1 c^* Z e^{j(\theta_1-\theta_2^*)}, & -j\omega_1 c^* Z e^{-j(\theta_1+\theta_2^*)}, & (1+\sigma_2)e^{-2j\theta_2^*}, & -\sigma_2 \\ j\omega_1 c^* Z e^{j(\theta_1+\theta_2^*)}, & -j\omega_1 c^* Z e^{j(\theta_2^*-\theta_1)}, & \sigma_2, & (1-\sigma_2)e^{2j\theta_2^*} \end{bmatrix}. \tag{15}$$

A little manipulation shows that

$$\tilde{T} \Omega^{-1} T = \Omega^{-1} \tag{16}$$

where

$$\Omega^{-1} = \begin{bmatrix} \frac{1}{\omega_1} & 0 & 0 & 0 \\ 0 & -\frac{1}{\omega_1} & 0 & 0 \\ 0 & 0 & -\frac{1}{\omega_2} & 0 \\ 0 & 0 & 0 & \frac{1}{\omega_2} \end{bmatrix}. \tag{17}$$

Since $T_\theta = I_\theta^{-1} T$,

$$\tilde{T}_\theta = \widetilde{I_\theta^{-1} T} = \tilde{T} \tilde{I}_\theta^{-1} = \tilde{T} I_\theta. \tag{18}$$

[2] The T-matrix of the conjugate circuit is closely related to the notion of the inverse of the conjugate operator in functional analysis. For the conjugate operator, the reader is referred to A. E. Taylor, "Introduction to Funcational Analysis," John Wiley and Sons, Inc., New York, N. Y., ch. 4, 1958. The waveguide directly related to the conjugate operator has been discussed in A. D. Bresler, G. H. Joshi, and N. Marcuvitz, "Orthogonality properties for modes in passive and active uniform wave guides," *J. Appl. Phys.*, vol. 29, pp. 794–799; May, 1958.

From (16) and (18), we obtain

$$\tilde{T}_\theta \Omega^{-1} T_\theta = \tilde{T} I_\theta \Omega^{-1} I_\theta^{-1} T = \tilde{T} \Omega^{-1} T = \Omega^{-1};$$

that is,

$$\tilde{T}_\theta \Omega^{-1} T_\theta = \Omega^{-1}. \tag{19}$$

This is the relation which corresponds to (27) in the previous paper.[1]

If A_l is an eigenvector of the operator T_θ,

$$(T_\theta - \lambda_l I) A_l = 0. \tag{20}$$

Taking the complex conjugate transpose of the corresponding relation for the conjugate circuit, we have

$$\tilde{A}_k (\tilde{T}_\theta - \tilde{\lambda}_k I) = 0 \tag{21}$$

where $\tilde{A}_k$ is the complex conjugate transpose of the eigenvector of the conjugate circuit, and $\tilde{\lambda}_k$ is the complex conjugate of the eigenvalue.

From (19)–(21), the following theorems can be derived in a similar way as for the lossless case (see Appendix).

Theorem 1: There is always an eigenvalue λ_k of T_θ, corresponding to an eigenvalue $\tilde{\lambda}_k$ of $\tilde{T}_\theta$, such that $\lambda_k = 1/\tilde{\lambda}_k$.

Theorem 2: If $l \neq k$, then A_l and $\tilde{A}_k$ are orthogonal in the sense that

$$\tilde{A}_k \Omega^{-1} A_l = 0. \tag{22}$$

Theorem 3: The corresponding modes $\tilde{A}_k$ and A_k are not orthogonal in the above sense; *i.e.*,

$$\tilde{A}_k \Omega^{-1} A_k \neq 0. \tag{23}$$

In the case of lossless amplifiers, the conjugate and original circuits are identical in the pass band of ω_1 and ω_2. Thus, $\tilde{A}_k$ is the complex conjugate transpose of the eigenvector of T_θ with the eigenvalue $1/\lambda_k^*$, leading to the same result obtained in the previous paper.

For the sake of convenience, we normalize the mode amplitudes so that we can rewrite (22) and (23) in the form

$$\tilde{A}_k \Omega^{-1} A_l = \delta_{kl} \tag{24}$$

where δ_{kl} is the Kronecker delta.

An arbitrary vector can be expressed as a sum of eigenvectors:

$$A = \sum \alpha_k A_k.$$

Multiplying by $\tilde{A}_k \Omega^{-1}$ from the left and using (24), we have

$$\alpha_k = \tilde{A}_k \Omega^{-1} A.$$

Therefore, for an arbitrary vector A, we obtain

$$A = \sum A_k (\tilde{A}_k \Omega^{-1} A). \tag{25}$$

Multiplying by T_θ^n from the left, we have

$$T_\theta^n A = \sum \lambda_k^n A_k (\tilde{A}_k \Omega^{-1} A). \tag{26}$$

Since A is an arbitrary vector, then

$$T_\theta^n = \sum \lambda_k^n A_k \cdot \tilde{A}_k \Omega^{-1}. \tag{27}$$

If $n = 1$,

$$T_\theta = \sum \lambda_k A_k \cdot \tilde{A}_k \Omega^{-1}. \tag{28}$$

This is the spectral representation of the operator T_θ.

Power Gain of the Amplifier

To obtain the expressions for the power gain of the amplifier, we need the solutions of the eigenvalue problem of the operator T_θ. We assume that

$$\theta_p = \theta_1 + \theta_2^* + \Delta\theta. \tag{29}$$

All the eigenvalues and the corresponding eigenvectors can be obtained by the method of perturbation. To the first order of approximation, they are

$$\lambda_1 = \left(1 - \frac{\sigma_1 + \sigma_2}{2} + \delta\right) e^{j(\theta_2^* - \theta_1)}, \quad \lambda_2 = (1 + \sigma_1 + j\Delta\theta) e^{j(3\theta_1 + \theta_2^*)}$$

$$A_1 = \begin{bmatrix} \sqrt{\dfrac{\omega_1}{2}} \\ \dfrac{-j}{2 \sin 2\theta_1} (\delta + \epsilon - \sigma_1) \sqrt{\dfrac{\omega_1}{2}} \\ j \dfrac{\delta + \epsilon}{\delta_0} \sqrt{\dfrac{\omega_2}{2}} \dfrac{c^*}{|c|} e^{-j\theta_p} \\ \dfrac{-1}{2 \sin 2\theta_2^*} (\delta - \epsilon - \sigma_2) \dfrac{\delta + \epsilon}{\delta_0} \sqrt{\dfrac{\omega_2}{2}} \dfrac{c^*}{|c|} e^{-j\theta_p} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \dfrac{j\sigma_1}{2 \sin 2\theta_1} \sqrt{\omega_1} \\ \sqrt{\omega_1} \\ \dfrac{\delta_0}{2 \sin 2\theta_1} \sqrt{\omega_2} \dfrac{c^*}{|c|} e^{-j\theta_p} \\ \dfrac{-\delta_0}{2 \sin 2(\theta_1 + \theta_2^*)} \sqrt{\omega_2} \dfrac{c^*}{|c|} e^{-j\theta_p} \end{bmatrix},$$

$$\lambda_3 = \left(1 - \frac{\sigma_1 + \sigma_2}{2} - \delta\right) e^{j(\theta_2^* - \theta_1)}, \qquad \lambda_4 = (1 + \sigma_2 - j\Delta\theta) e^{-j(\theta_1 + 3\theta_2^*)},$$

$$A_3 = \begin{bmatrix} \sqrt{\frac{\omega_1}{2}} \\ \frac{j}{2 \sin 2\theta_1} (\delta - \epsilon + \sigma_1) \sqrt{\frac{\omega_1}{2}} \\ -j \frac{\delta - \epsilon}{\delta_0} \sqrt{\frac{\omega_2}{2}} \frac{c^*}{|c|} e^{-j\theta_p} \\ \frac{-1}{2 \sin 2\theta_2^*} (\delta + \epsilon + \sigma_2) \frac{\delta - \epsilon}{\delta_0} \sqrt{\frac{\omega_2}{2}} \frac{c^*}{|c|} e^{-j\theta_p} \end{bmatrix}, \qquad A_4 = \begin{bmatrix} \frac{\delta_0}{2 \sin 2\theta_2^*} \sqrt{\omega_1} \frac{c}{|c|} e^{j\theta_p} \\ \frac{-\delta_0}{2 \sin 2(\theta_1 + \theta_2^*)} \sqrt{\omega_1} \frac{c}{|c|} e^{j\theta_p} \\ \frac{-j\sigma_2}{2 \sin 2\theta_2^*} \sqrt{\omega_2} \\ \sqrt{\omega_2} \end{bmatrix} \tag{30}$$

where

$$\delta_0 = \sqrt{\omega_1 \omega_2} |c| Z$$

$$\epsilon = \left(\frac{\sigma_1 - \sigma_2}{2} - j\Delta\theta\right)$$

$$\delta = \sqrt{\delta_0^2 + \epsilon^2}. \tag{31}$$

The first and third modes are the forward waves traveling from the input to the output. $-(\sigma_1+\sigma_2)/2$ in the eigenvalues roughly represents the attenuation due to the spreading resistance. On the other hand, δ roughly represents the amplification due to the variable capacitor. If the real part of $(\sigma_1+\sigma_2)/2$ is larger than the real part of δ, no net amplification takes place. It is worth noting that the components of the first and third modes are considerably influenced by the spreading resistance if $\sigma_1 \neq \sigma_2$.

The second and fourth modes are the backward waves, which remain almost unaffected. The magnitude of the eigenvalues are greater than unity, which means that these backward waves attenuate with propagation.

Replacing θ_1, θ_2^*, $\Delta\theta$, σ_1, σ_2 and Z in (30) by θ_1^*, θ_2, $\Delta\theta^*$, $-\sigma_1^*$, $-\sigma_2^*$ and Z^*, respectively, and interchanging the subscripts $k=1$ and 3, we have the eigenvectors of the conjugate circuit. The complex conjugate transposes of the eigenvectors multiplied by appropriate normalization constants give $\tilde{A}_k$'s:

$$\tilde{A}_1 = \left[\frac{\delta_0^2}{\delta(\delta + \epsilon)} \sqrt{\frac{\omega_1}{2}}, \frac{-j}{2 \sin 2\theta_1} \frac{\delta_0^2}{\delta(\delta + \epsilon)} (\delta + \epsilon - \sigma_1) \sqrt{\frac{\omega_1}{2}}, j \frac{\delta_0}{\delta} \sqrt{\frac{\omega_2}{2}} \frac{c}{|c|} e^{j\theta_p}, \frac{-1}{2 \sin 2\theta_2^*} \frac{\delta_0}{\delta} (\delta - \epsilon - \sigma_2) \sqrt{\frac{\omega_2}{2}} \frac{c}{|c|} e^{j\theta_p}\right],$$

$$\tilde{A}_2 = \left[\frac{-j\sigma_1}{2 \sin 2\theta_1} \sqrt{\omega_1}, -\sqrt{\omega_1}, \frac{-\delta_0}{2 \sin 2\theta_1} \sqrt{\omega_2} \frac{c}{|c|} e^{j\theta_p}, \frac{\delta_0}{2 \sin 2(\theta_1 + \theta_2^*)} \sqrt{\omega_2} \frac{c}{|c|} e^{j\theta_p}\right],$$

$$\tilde{A}_3 = \left[\frac{\delta_0^2}{\delta(\delta - \epsilon)} \sqrt{\frac{\omega_1}{2}}, \frac{j}{2 \sin 2\theta_1} \frac{\delta_0^2}{\delta(\delta - \epsilon)} (\delta - \epsilon + \sigma_1) \sqrt{\frac{\omega_1}{2}}, -j \frac{\delta_0}{\delta} \sqrt{\frac{\omega_2}{2}} \frac{c}{|c|} e^{j\theta_p}, \frac{-1}{2 \sin 2\theta_2^*} \frac{\delta_0}{\delta} (\delta + \epsilon + \sigma_2) \sqrt{\frac{\omega_2}{2}} \frac{c}{|c|} e^{j\theta_p}\right],$$

$$\tilde{A}_4 = \left[\frac{\delta_0}{2 \sin 2\theta_2^*} \sqrt{\omega_1} \frac{c^*}{|c|} e^{-j\theta_p}, \frac{-\delta_0}{2 \sin 2(\theta_1 + \theta_2^*)} \sqrt{\omega_1} \frac{c^*}{|c|} e^{-j\theta_p}, \frac{-j\sigma_2}{2 \sin 2\theta_2^*} \sqrt{\omega_2}, \sqrt{\omega_2}\right]. \tag{32}$$

Eqs. (30) and (32) satisfy the orthogonality theorems (24) to the first order of approximation.

Substituting (30) and (32) in (26) and then using (13), we obtain four equations.

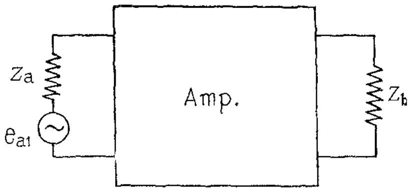

Fig. 3—Input and output conditions of the amplifier.

$$b_{i1} = e^{-jn\theta_p}\left\{\left(\frac{\lambda_1^n}{2}\,\frac{\delta-\epsilon}{\delta} + \frac{\lambda_3^n}{2}\,\frac{\delta+\epsilon}{\delta}\right)a_{i1} + \left(\frac{\lambda_1^n}{2} - \frac{\lambda_3^n}{2}\right)(-j)\,\frac{\delta_0}{\delta}\sqrt{\frac{\omega_1}{\omega_2}}\,\frac{c}{|c|}\,e^{j\theta_p}a_{i2}^* + \cdots\right\},$$

$$b_{r1} = e^{-jn\theta_p}\{\lambda_2^n a_{r1} + \cdots\},$$

$$b_{i2}^* = e^{jn\theta_p}\left\{\left(\frac{\lambda_1^n}{2} - \frac{\lambda_3^n}{2}\right)j\,\frac{\delta_0}{\delta}\sqrt{\frac{\omega_2}{\omega_1}}\,\frac{c^*}{|c|}\,e^{-j\theta_p}a_{i1} + \left(\frac{\lambda_1^n}{2}\,\frac{\delta+\epsilon}{\delta} + \frac{\lambda_3^n}{2}\,\frac{\delta-\epsilon}{\delta}\right)a_{i2}^* + \cdots\right\},$$

$$b_{r2}^* = e^{jn\theta_p}\{\lambda_4^n a_{r2}^* + \cdots\}. \qquad (33)$$

From the input and output conditions shown in Fig. 3, we have

$$a_{i1} = \frac{\sqrt{Z_{01}}}{Z_{01}+Z_{a1}}\,e_{a1} + \Gamma_{a1}a_{r1}, \qquad a_{i2}^* = \Gamma_{a2}^* a_{r2}^*$$

$$b_{r1} = \Gamma_{b1}b_{i1}, \qquad b_{r2}^* = \Gamma_{b2}^* b_{i2}^* \qquad (34)$$

where the Γ's are the reflection coefficients:

$$\Gamma_a = \frac{Z_a - Z_0}{Z_a + Z_0}, \qquad \Gamma_b = \frac{Z_b - Z_0}{Z_b + Z_0}. \qquad (35)$$

There are altogether eight equations with eight unknown quantities a_{i1}, a_{i2}^*, $\cdots$, b_{r1}, b_{r2}^*. The simultaneous equations can be solved by the standard method of algebra. Neglecting the higher-order terms of δ and of the reflection coefficients, we have, for example,

$$b_{i1} \doteqdot \frac{\left(\frac{\lambda_1^n+\lambda_3^n}{2} - \frac{\epsilon}{\delta}\,\frac{\lambda_1^n-\lambda_3^n}{2}\right)e^{-jn\theta_p}\,\frac{\sqrt{Z_{01}}}{Z_{01}+Z_{a1}}\,e_{a1}}{1 - \left(\frac{\lambda_1^n+\lambda_3^n}{2} - \frac{\epsilon}{\delta}\,\frac{\lambda_1^n-\lambda_3^n}{2}\right)\frac{\Gamma_{a1}\Gamma_{b1}}{\lambda_2^n} - \left(\frac{\lambda_1^n+\lambda_3^n}{2} + \frac{\epsilon}{\delta}\,\frac{\lambda_1^n-\lambda_3^n}{2}\right)\frac{\Gamma_{a2}^*\Gamma_{b2}^*}{\lambda_4^n}}. \qquad (36)$$

We define the power gain G of the amplifier to be the ratio of the power dissipated in the load at ω_1 to the available power from the generator:

$$G = |b_{i1}|^2\,\mathrm{Re}\left\{\frac{Z_{01}}{|Z_{01}|}(1 - |\Gamma_{b1}|^2 + \Gamma_{b1} - \Gamma_{b1}^*)\right\} \Big/ \frac{|e_{a1}|^2}{4\,\mathrm{Re}\,Z_{a1}}. \qquad (37)$$

Substituting (36) in (37), we can obtain the power gain in its explicit form. For simplicity, we neglect the imaginary part of θ_1, θ_2, δ and Z_{01}, as well as the difference between σ_1 and σ_2 ($\sigma=\sigma_1=\sigma_2$). Furthermore, if the Γ's are all zero, (37) becomes

$$G_0 = \left\{\cosh^2 n\delta + \left(\frac{\Delta\theta}{\delta}\right)^2 \sinh^2 n\delta\right\} e^{-2n\sigma}. \qquad (38)$$

If $\Gamma_{a1}\Gamma_{b1}$ and $\Gamma_{a2}^*\Gamma_{b2}^*$ are small but not zero,

$$G = \frac{G_0\,\frac{4\,\mathrm{Re}\,Z_{a1}}{|Z_{01}+Z_{a1}|^2}\,\mathrm{Re}\,\{Z_{01}(1-|\Gamma_{b1}|^2+\Gamma_{b1}-\Gamma_{b1}^*)\}}{(1-\sqrt{G_0}\,\mathrm{Re}\,R)^2 + (\sqrt{G_0}\,\mathrm{Im}\,R)^2} \qquad (39)$$

where

$$R = e^{-n\sigma}\{e^{-jn(4\theta_1+\Delta\theta)}\Gamma_{a1}\Gamma_{b1}e^{j\phi} + e^{jn(4\theta_2^*+\Delta\theta)}\Gamma_{a2}^*\Gamma_{b2}^*e^{-j\phi}\} \qquad (40)$$

$$\tan\phi = \frac{\Delta\theta}{\delta}\tanh n\delta. \qquad (41)$$

For a numerical example, let

$$Z_{01}' = Z_{02}' = 30\Omega, \quad |c| = 3.5\mu\mu F, \quad c_0 = 5\,\mu\mu F$$

$$\omega_1 \doteqdot \omega_2 \doteqdot 670 \text{ mc}, \quad r = 1\Omega, \quad n = 16.$$

Then, from (11) and (31), we obtain

$$\delta_0 = 0.11 \qquad \sigma = 0.006.$$

Inserting these values in (38), we get

$$G_0 = 8.7 \text{ db} \qquad \text{for } \Delta\theta = 0$$

$$G_0 = 5.7 \text{ db} \qquad \text{for } \Delta\theta = 6^\circ$$

which can be compared with the second experimental result by Engelbrecht.[3] If $|R| = 0.1$, because of the de-

[3] R. S. Engelbrecht, "Nonlinear-Reactance (Parametric) Traveling-Wave Amplifiers for UHF," Presented at the 1959 Solid-State Circuits Conf., Philadelphia, Pa., February, 1959.

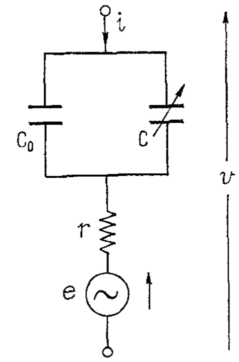

Fig. 4—Equivalent circuit of semiconductor diode with noise voltages.

nominator of (39), G varies from 6.6 db to 11.5 db, when $\Delta\theta = 0$, depending on the phase of R.

It should be recognized that (38) and (39) are the results of a three-frequency analysis. If the other sidebands are capable of propagating, the gain may be reduced.

Noise Figure Calculation

For the calculation of the noise figure, we must consider the noise voltages e_1 and e_2 due to the spreading resistance of the semiconductor diode. On the other hand, for simplicity, we neglect the losses, hence the noise generated in the two terminal pair networks (a) and (b) in Fig. 2.

The equivalent circuit of the diode is shown in Fig. 4. Replacing v_1 and v_2^* in (2) by $v_1 - e_1$ and $v_2^* - e_2^*$ respectively, we have the matrix representation of this equivalent circuit. Using this representation in a similar way as for (8), we get the relation

$$B = TA + KN \tag{42}$$

between the input vector A and the output vector B of the basic section of the amplifier, where T is given by (10) and

$$N = \begin{bmatrix} e_1 \\ e_2^* \end{bmatrix}.$$

Next, we shall consider the amplifier with n sections. For the mth section, c is replaced by $ce^{-2(m-1)j\theta_p}$ and K becomes

$$I_\theta^{m-1} K J_\theta^{-m+1}$$

where

$$J_\theta = \begin{bmatrix} e^{-j\theta_p} & 0 \\ 0 & e^{j\theta_p} \end{bmatrix}. \tag{44}$$

Therefore, the output vector of the amplifier is

$$\begin{aligned} B &= (I_\theta^{n-1} T I_\theta^{-n+1})(I_\theta^{n-2} T I_\theta^{-n+2}) \cdots (I_\theta T I_\theta^{-1}) T A \\ &\quad + \sum_{m=1}^{n} (I_\theta^{n-1} T I_\theta^{-n+1})(I_\theta^{n-2} T I_\theta^{-n+2}) \cdots \\ &\qquad \cdot (I_\theta^{m} T I_\theta^{-m}) I_\theta^{m-1} K J_\theta^{-m+1} N_m \\ &= I_\theta^n T_\theta^n A + \sum_{m=1}^{n} I_\theta^n T_\theta^{n-m} I_\theta^{-1} K J_\theta^{-m+1} N_m, \end{aligned} \tag{45}$$

where A is the input vector of the amplifier and N_m represents the matrix N with the noise voltages e_{m1} and e_{m2}^* of the mth diode. Using (27), we rewrite (45) in the form

$$\begin{aligned} B = I_\theta^n \Big\{ &\sum_k \lambda_k^n A_k (\tilde{A}_k \Omega^{-1} A) \\ &+ \sum_{m=1}^{n} \sum_k \lambda_k^{n-m} A_k (\tilde{A}_k \Omega^{-1} I_\theta^{-1} K J_\theta^{-m+1} N_m) \Big\}. \end{aligned} \tag{46}$$

$$K = \begin{bmatrix} \frac{1}{2}\sqrt{Z_{01}'}e^{-j\theta_1}\left\{j\omega_1 c_0 + r\left(\omega_1^2 c_0^2 - \omega_1\omega_2 \frac{|c|^2}{4}\right)\right\}, & \frac{1}{2}\sqrt{Z_{01}'}e^{-j\theta_1} j\omega_1 \frac{c}{2}\{1 + j(\omega_2 - \omega_1)c_0 r\} \\ -\frac{1}{2}\sqrt{Z_{01}'}e^{j\theta_1}\left\{j\omega_1 c_0 + r\left(\omega_1^2 c_0^2 - \omega_1\omega_2 \frac{|c|^2}{4}\right)\right\}, & -\frac{1}{2}\sqrt{Z_{01}'}e^{j\theta_1} j\omega_1 \frac{c}{2}\{1 + j(\omega_2 - \omega_1)c_0 r\} \\ -\frac{1}{2}\sqrt{Z_{02}'^*}e^{j\theta_2^*} j\omega_2 \frac{c^*}{2}\{1 + j(\omega_2 - \omega_1)c_0 r\}, & -\frac{1}{2}\sqrt{Z_{02}'^*}e^{j\theta_2^*}\left\{j\omega_2 c_0 - r\left(\omega_2^2 c_0^2 - \omega_1\omega_2 \frac{|c|^2}{4}\right)\right\} \\ \frac{1}{2}\sqrt{Z_{02}'^*}e^{-j\theta_2^*} j\omega_2 \frac{c^*}{2}\{1 + j(\omega_2 - \omega_1)c_0 r\}, & \frac{1}{2}\sqrt{Z_{02}'^*}e^{-j\theta_2^*}\left\{j\omega_2 c_0 - r\left(\omega_2^2 c_0^2 - \omega_1\omega_2 \frac{|c|^2}{4}\right)\right\} \end{bmatrix}. \tag{43}$$

Substituting (30) and (32) in (46), we obtain

$$
\begin{aligned}
b_{i1} \doteqdot e^{-jn\theta_p}\Bigg[&\left(\frac{\lambda_1^n+\lambda_3^n}{2}-\frac{\epsilon}{\delta}\,\frac{\lambda_1^n-\lambda_3^n}{2}\right)a_{i1}\\
&-\frac{\lambda_1^n-\lambda_3^n}{2}\,j\,\frac{\delta_0}{\delta}\sqrt{\frac{\omega_1}{\omega_2}}\,\frac{c}{|c|}\,e^{j\theta_p}a_{i2}^*+\cdots\\
&+\sum_{m=1}^{n}\Bigg\{\left(\frac{\lambda_1^{n-m}+\lambda_3^{n-m}}{2}-\frac{\epsilon}{\delta}\,\frac{\lambda_1^{n-m}-\lambda_3^{n-m}}{2}\right)m_{i1}\\
&-\frac{\lambda_1^{n-m}-\lambda_3^{n-m}}{2}\,j\,\frac{\delta_0}{\delta}\sqrt{\frac{\omega_1}{\omega_2}}\,\frac{c}{|c|}\,e^{j\theta_p}m_{i2}^*\Bigg\}\\
&+\cdots\Bigg]
\end{aligned}
\tag{47}
$$

with similar equations for the other components, where

$$
I_\theta^{-1}KJ_\theta^{-m+1}N_m=\begin{bmatrix} m_{i1}\\ m_{r1}\\ m_{i2}^*\\ m_{r2}^*\end{bmatrix}\doteqdot\begin{bmatrix}
\frac{1}{2}j\omega_1\sqrt{Z_{01}'}e^{-j\theta_1}\left\{c_0e^{jm\theta_p}e_{m_1}+\frac{c}{2}e^{(2-m)j\theta_p}e_{m_2}^*\right\}\\
-\frac{1}{2}j\omega_1\sqrt{Z_{01}'}e^{j\theta_1}\left\{c_0e^{jm\theta_p}e_{m_1}+\frac{c}{2}e^{(2-m)j\theta_p}e_{m_2}^*\right\}\\
-\frac{1}{2}j\omega_2\sqrt{Z_{02}'^*}e^{j\theta_2}\left\{\frac{c^*}{2}e^{(m-2)j\theta_p}e_{m_1}+c_0e^{-jm\theta_p}e_{m_2}^*\right\}\\
\frac{1}{2}j\omega_2\sqrt{Z_{02}'^*}e^{-j\theta_2}\left\{\frac{c^*}{2}e^{(m-2)j\theta_p}e_{m_1}+c_0e^{-jm\theta_p}e_{m_2}^*\right\}
\end{bmatrix}.
\tag{48}
$$

For simplicity, hereafter we shall confine ourselves to the case in which we obtained (38): $\sigma=\sigma_1=\sigma_2$ and the reflection coefficients are all zero. Then, from (47), the noise output power becomes

$$
\begin{aligned}
N_0=\overline{b_{c1}b_{c1}^*}\doteqdot\ &G_0kT\Delta f\\
&+\left(\frac{|\lambda_1|^n-|\lambda_3|^n}{2}\right)^2\frac{\delta_0^2}{\delta^2}\,\frac{\omega_1}{\omega_2}\,kT\Delta f\\
&+\ kT_r\Delta fr\left[\frac{1}{4}\left(\frac{|-|\lambda_1|^{2n}}{|-|\lambda_1|^2}+\frac{|-|\lambda_3|^{2n}}{|-|\lambda_3|^2}\right)\right.\\
&\cdot\left\{\frac{\delta_0^2}{\delta^2}\left(c_0^2+\frac{|c|^2}{4}\right)(\omega_1^2Z_{01}'+\omega_1\omega_2Z_{02}')\right.\\
&\left.+\frac{2\Delta\theta\delta_0}{\delta^2}\,c_0|c|\,\omega_1\omega_2\sqrt{\frac{\omega_1}{\omega_2}}\sqrt{Z_{01}'Z_{02}'}\right\}\\
&+\frac{1}{2}\,\frac{|-|\lambda_1\lambda_3|^n}{|-|\lambda_1\lambda_3|}\left\{\left(c_0^2+\frac{|c|^2}{4}\right)\right.\\
&\cdot\left(\omega_1^2Z_{01}'-\frac{\Delta\theta^2}{\delta^2}\omega_1^2Z_{01}'-\frac{\delta_0^2}{\delta^2}\omega_1\omega_2Z_{02}'\right)\\
&\left.\left.-\frac{2\Delta\theta\delta_0}{\delta^2}c_0|c|\,\omega_1\omega_2\sqrt{\frac{\omega_1}{\omega_2}}\sqrt{Z_{01}'Z_{02}'}\right\}\right]
\end{aligned}
\tag{49}
$$

where

k = Boltzmann's constant,
Δf = the noise band width of the amplifier,
T = standard noise temperature, and
T_r = the equivalent noise temperature of the spreading resistance.

If $|\lambda_1|^{2n}\gg|\lambda_3|^{2n}$, as is usually the case, (49) becomes

$$
\begin{aligned}
N_0=\ &G_0kT\Delta f+\frac{|\lambda_1|^{2n}}{\lambda}\,\frac{\delta_0^2}{\delta^2}\,\frac{\omega_1}{\omega_2}\,kT\Delta f+kT_r\Delta fr\,\frac{|\lambda_1|^{2n}}{4}\\
&\cdot\frac{1}{2(\delta-\sigma)}\left\{\frac{\delta_0^2}{\delta^2}\left(c_0^2+\frac{|c|^2}{4}\right)(\omega_1^2Z_{01}'\times\omega_1\omega_2Z_{02}')\right.\\
&\left.+\frac{2\Delta\theta\delta_0}{\delta^2}c_0|c|\,\omega_1\omega_2\sqrt{\frac{\omega_1}{\omega_2}}\sqrt{Z_{01}'Z_{02}'}\right\}.
\end{aligned}
\tag{50}
$$

In this case,

$$
G_0\doteqdot\frac{1}{4}\,|\lambda_1|^{2n}\,\frac{\delta_0^2}{\delta^2}.
$$

Dividing N_0 by $G_0kT\Delta f$, we obtain the noise figure F_0.

$$
\begin{aligned}
F_0=1+\frac{\omega_1}{\omega_2}+\frac{T_r}{T}\,\frac{rZ_{01}'\omega_1^2c_0^2}{2(\delta-\sigma)}&\left\{\left(1+\frac{\omega_2}{\omega_1}\,\frac{Z_{02}'}{Z_{01}'}\right)\right.\\
&\left.\cdot\left(1+\frac{|c|^2}{4c_0^2}\right)+\frac{2\Delta\theta}{\delta_0}\,\frac{|c|}{c_0}\sqrt{\frac{\omega_2}{\omega_1}}\sqrt{\frac{Z_{02}'}{Z_{01}'}}\right\}.
\end{aligned}
\tag{51}
$$

For a numerical example, we use the same values described in the previous section, and assume that $T_r=T$. Then (54) gives

F_0 = 3.3 db for $\Delta\theta=0$

F_0 = 4.5 db for $\Delta\theta=6°$.

Appendix

Theorems 1, 2, and 3 will be proved.

From (21), the determinant of $(\tilde{T}_\theta-\lambda_kI)$ vanishes:

$$
\det(\tilde{T}_\theta-\lambda_kI)=0.
\tag{52}
$$

Since det $T_\theta \neq 0$, $\tilde{\lambda}_k \neq 0$. From these relations, we have

$$\det(\tilde{T}_\theta - \tilde{\lambda}_k I)\det(\Omega^{-1}T_\theta) = \det(\tilde{T}_\theta\Omega^{-1}T_\theta - \tilde{\lambda}_k\Omega^{-1}T_\theta)$$
$$= \det(\Omega^{-1} - \tilde{\lambda}_k\Omega^{-1}T_\theta)$$
$$= \det(\Omega^{-1}\tilde{\lambda}_k)\det\left(\frac{1}{\tilde{\lambda}_k}I - T_\theta\right) = 0.$$

The final result is

$$\det\left(T_\theta - \frac{1}{\tilde{\lambda}_k}I\right) = 0. \qquad (53)$$

From (53), it follows that there is always an eigenvalue $1/\tilde{\lambda}_k$ of T_θ, corresponding to an eigenvalue $\tilde{\lambda}_k$ to $\tilde{T}_\theta$, (Theorem 1).

We shall use the same subscript for the corresponding solutions of the eigenvalue problems of the two circuits:

$$\lambda_k = \frac{1}{\tilde{\lambda}_k}. \qquad (54)$$

Multiplying (21) by $\Omega^{-1}T_\theta A_l$ from the right and using (19) and (20), we obtain

$$\left(\lambda_l - \frac{1}{\tilde{\lambda}_k}\right)\tilde{A}_k\Omega^{-1}A_l = 0. \qquad (55)$$

If $\lambda_l \neq 1/\tilde{\lambda}_k$, (55) shows that $\tilde{A}_k\Omega^{-1}A_l = 0$. In the non-degenerate case, $\lambda_l \neq 1/\tilde{\lambda}_k$ for $k \neq l$. Thus, we obtain the desired orthogonality relation (Theorem 2):

$$\tilde{A}_k\Omega^{-1}A_l = 0, \qquad k \neq l. \qquad (56)$$

In the degenerate case, $k \neq l$ does not necessarily mean that $\lambda_l \neq 1/\tilde{\lambda}_k$. However, we are justified in assuming (56), for it is always possible to introduce the degenerate eigenvectors in such a way as to secure the orthogonality.

Next, we expand $\Omega\tilde{A}_k^+$ by the eigenvectors A_l, where the symbol $^+$ indicates the complex conjugate transpose:

$$\Omega\tilde{A}_k^+ = \sum \alpha_l A_l.$$

Multiplying by $\tilde{A}_k\Omega^{-1}$ from the left and using (56), we have

$$\tilde{A}_k\tilde{A}_k^+ = \alpha_k\tilde{A}_k\Omega^{-1}A_k.$$

Since $\tilde{A}_k \neq 0$, the left hand side of the above equation is not zero. Thus we conclude that (Theorem 3):

$$\tilde{A}_k\Omega^{-1}A_k \neq 0. \qquad (57)$$

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Action of a Progressive Disturbance on a Guided Electromagnetic Wave*

J. C. SIMON†

I. INTRODUCTION

A.

A PROBLEM often encountered in wave physics concerns the interaction of various types of waves, and the energy transfer from one wave to another.

In the particular case of waves of the same nature, "modes" can be distinguished in such a way that a wave can be represented as a sum of these modes. Their essential character is that the energy associated with each does not vary with time. It is also said that these modes are not "coupled." This, for instance, is the case of waves guided in an electric waveguide, of mechanical vibration in a bar, and of energy levels in quantum physics.

Although this possibility of decomposition in "normal modes" corresponds to particular physical conditions, it has made it possible to deduce general notions of a fundamental character essential to the physicist. In the most general case, the normal modes are said to be coupled that is the energy passes from one to the other, so much that this decomposition into normal modes appears to be indispensable in deducing physical concepts.

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